

CHARACTERISTIC PROPERTIES OF NEHARI FUNCTIONS

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ABSTRACT. Let N be the set of all meromorphic functions f defined in the unit disc D that satisfy Nehari's univalence criterion $(1 - |z|^2)^2 |Sf(z)| \leq 2$. In this paper we investigate certain properties of the class N . We obtain sharp estimates for the spherical distortion, and also a two-point distortion theorem that actually characterizes the set N . Finally, we study some aspects of the boundary behavior of Nehari functions, and obtain results that indicate how such maps can fail to map D onto a quasidisc.

1. INTRODUCTION

Let f be analytic in the unit disc D and let $Sf = (f''/f')' - (1/2)(f''/f')^2$ be its Schwarzian derivative. In 1949 Nehari showed that if

$$|Sf(z)| \leq \frac{2}{(1 - |z|^2)^2} \quad (1.1)$$

for all $z \in D$ then f is univalent [11]. A necessary condition for univalence is obtained by replacing the 2 with a 6 in the numerator of (1.1). It was proved by Kraus in 1932, [9], and rediscovered later by Nehari.

Let N be the set of all meromorphic functions satisfying (1.1). This *Nehari Class* was formally introduced and extensively studied in [5]. In the present paper we will make use of several results from [5] and also earlier papers, and it is our purpose here to investigate further properties of Nehari maps.

We shall consider functions in N normalized in two different ways. In the first one,

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \cdots, \quad (1.2)$$

while in the second, we let

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0. \quad (1.3)$$

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Both normalizations are achieved by composing f from the left with suitable Möbius transformations. This leaves (1.1) invariant. The second normalization gives rise to the class N_0 , and according to a result in [3] if f satisfies (1.1) and (1.3) then it has no poles. In fact, such a function will either be a rotation of the logarithm

$$L(z) = \frac{1}{2} \log \frac{1+z}{1-z}, \quad (1.4)$$

or else it will be bounded. The function L has

$$SL(z) = \frac{2}{(1-z^2)^2} \quad (1.5)$$

and plays a very important role and is extremal for many problems in the class N .

There is a classical connection between the Schwarzian and second order linear differential equations. If $Sf = 2p$ and $u = (f')^{-1/2}$ then

$$u'' + pu = 0. \quad (1.6)$$

Conversely, if u_1, u_2 are linearly independent solutions of (1.6) and $f = u_1/u_2$ then $Sf = 2p$.

Much of the work in [3] is based on applying comparison theorems for solutions of differential equations to obtain bounds on f and f' . For example, if $f \in N_0$ then $u = (f')^{-1/2}$ satisfies the initial conditions $u(0) = 1, u'(0) = 0$, and it was shown that

$$n(|z|) \leq |f(z)| \leq L(|z|), \quad (1.7)$$

and

$$n'(|z|) \leq |f'(z)| \leq L'(|z|), \quad (1.8)$$

where

$$n(z) = \frac{1}{\sqrt{2}} \frac{(1+z)^{\sqrt{2}} - (1-z)^{\sqrt{2}}}{(1+z)^{\sqrt{2}} + (1-z)^{\sqrt{2}}}.$$

The function n belongs to N_0 and has $Sn(z) = -2/(1-z^2)^2$.

The techniques of comparison allow one also to describe the cases of equality: if equality holds in (1.7) or (1.8) at a single $z_0 \neq 0$ then f must be a rotation of the corresponding extremal, n or L .

In Section 2 we shall consider equation (1.6) but with the dual initial condition, namely, $u(0) = 0, u'(0) = 1$. In terms of the function f this means assuming the normalization (1.2). By considering $g = 1/f$,

$f \in N_0$, we will derive in this way sharp upper and lower bounds for the spherical distortion $|f'|/(1+|f|^2)$. We will also obtain a two-point distortion theorem that actually characterizes Nehari functions. This result can be viewed as an analogue of a theorem of Blatter that characterizes the set of all univalent functions in D [2].

One of the main results in [5] is the fact that, for a function in N , the image domain is a quasidisc as soon as it is a John domain. In other words, linear connectivity comes as a consequence of the John condition. Recall that if f is any univalent function then one of the many characterizations of John domains is that there exists a constant M such that for all $z \in D$

$$\text{diam}f(B(z)) \leq M(1 - |z|^2)|f'(z)|, \quad (1.9)$$

where

$$B(z) = \{w : |z| \leq |w| < 1, |\arg(w) - \arg(z)| \leq \pi(1 - |z|)\}. \quad (1.10)$$

For a detailed exposition of these concepts we refer the reader to [12, Chapter 5].

In Section 3 we will derive an estimate for $\text{diam}f(B(z))$ when $f \in N$, which will indicate how a Nehari domain can fail to be a quasidisc. Finally, in Section 4 we will be concerned with other ‘quasidisc like’ properties of Nehari domains, expressed in terms of f''/f' .

2. TWO-POINT DISTORTION AND CHARACTERIZATION

The starting point in this section is a comparison lemma. It is essentially contained in [3], and we include here a brief proof for the convenience of the reader.

Lemma 1: *Let $P = P(x) \geq 0$ be continuous for $x \in [0, 1)$ and suppose that the solution of*

$$v''(x) + P(x)v(x) = 0, \quad v(0) = 0, \quad v'(0) = 1 \quad (2.1)$$

is positive in the open interval $(0,1)$. Let w be solution of

$$w''(x) - P(x)w(x) = 0, \quad w(0) = 0, \quad w'(0) = 1. \quad (2.2)$$

If $p = p(z)$ is analytic in D and $|p(z)| \leq P(|z|)$ then the solution of

$$u'' + pu = 0, \quad u(0) = 0, \quad u'(0) = 1 \quad (2.3)$$

satisfies

$$v(|z|) \leq |u(z)| \leq w(|z|). \quad (2.4)$$

Proof: We consider u along rays starting from the origin, and without loss of generality, we may take the segment $[0, 1)$. Thus let $\varphi(x) = |u(x)|$ for $x \in [0, 1)$. At $x = 0$ the right hand derivative of φ exists and equals 1. Whenever $u(x) \neq 0$ then φ is smooth, and it is not difficult to show that

$$\varphi''(x) + |p(x)|\varphi(x) \geq 0.$$

Since the function v is positive in $(0, 1)$, it follows from the Sturm comparison theorem that

$$\varphi(x) \geq v(x)$$

for all $x \in [0, 1)$. This proves the lower bound in (2.4).

In order to establish the remaining inequality we turn (2.3) into the integral equation

$$u(z) = z - \int_0^z (z - \zeta)p(\zeta)u(\zeta)d\zeta.$$

Since $|p(z)| \leq P(|z|)$ it is a consequence of Lemma 8 in [6] that

$$|u(z)| \leq w(|z|).$$

This finishes the proof.

Theorem 1: *Let $f \in N_0$. Then*

$$\frac{n'(|z|)}{1 + n^2(|z|)} \leq \frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{L'(|z|)}{1 + L^2(|z|)}. \quad (2.5)$$

If equality holds in either inequality at a single $z_0 \neq 0$ then f is a rotation of the corresponding extremal.

Proof: Let $f \in N_0$ and let $g = 1/f$. Then g satisfies (1.1) and (1.2), hence for $u = (g')^{-1/2}$ one has

$$u'' + \left(\frac{1}{2}Sf\right)u = 0, \quad u(0) = 0, \quad u'(0) = 1.$$

With $P(x) = (1 - x^2)^{-2}$ the functions v, w of the lemma are given by

$$v(x) = \sqrt{\frac{L^2}{L'}}(x)$$

and

$$w(x) = \sqrt{\frac{n^2}{n'}}(x).$$

So the lemma yields

$$\frac{n'}{n^2}(|z|) \leq \left| \frac{f'}{f^2}(z) \right| \leq \frac{L'}{L^2}(|z|).$$

Hence, using in addition (1.8), we obtain

$$\frac{1 + |f(z)|^2}{|f'(z)|} = \frac{1}{|f'(z)|} + \frac{|f^2(z)|}{|f'(z)|} \geq \frac{1}{L'(|z|)} + \frac{L^2(|z|)}{L'(|z|)},$$

and similarly,

$$\frac{1 + |f(z)|^2}{|f'(z)|} \leq \frac{1}{n'(|z|)} + \frac{n^2(|z|)}{n'(|z|)}.$$

These two inequalities give (2.5).

Finally, if equality holds in (2.5) at some $z_0 \neq 0$ then it follows already from the case of equality in (1.8) that f must be a rotation of n or L . This finishes the proof.

If $f \in S$, the class of all univalent function in D with $f(0) = 0$, $f'(0) = 1$ then, as mentioned in the introduction, one has $|Sf(z)| \leq 6(1 - |z|^2)^{-2}$. Again by looking at $g = 1/f$ and $u = (g')^{-1/2}$ we can apply Lemma 1, but now only with the solution w because the corresponding function v has (infinitely many) zeroes [8, p. 492]. The function w arises from the Koebe function $k(z) = z/(1 - z)^2$, that is,

$$w(x) = \sqrt{\frac{k^2}{k'}}(x),$$

and we obtain in this fashion the sharp estimate

$$\left| \frac{f'}{f^2}(z) \right| \geq \frac{1 - |z|^2}{|z|^2}.$$

This inequality is equivalent to one established in 1919 by Löwner, namely that for functions g in the class Σ ,

$$|g'(\zeta)| \geq 1 - \frac{1}{|\zeta|^2}, \quad |\zeta| > 1.$$

It is interesting to note that in our proof we only use the fact that $(1 - |z|^2)^2 |Sf(z)| \leq 6$, rather than the univalence of f .

The next result characterizes Nehari functions in terms of a two-point distortion property. Let $d_h(z_1, z_2)$ be the hyperbolic distance between points in D .

Theorem 2: *Let f be meromorphic and locally univalent in D . Then*

$$(1 - |z|^2)^2 |Sf(z)| \leq 2 \quad (2.6)$$

for all $z \in D$ if and only if

$$(1 - |z_1|^2) |f'(z_1)| (1 - |z_2|^2) |f'(z_2)| d_h(z_1, z_2)^2 \leq |f(z_1) - f(z_2)|^2 \quad (2.7)$$

for all $z_1, z_2 \in D$. Furthermore, equality holds for $z_1 \neq z_2$ if and only if f is of the form $T \circ L \circ \tau$, where T is Möbius and τ an automorphism of D with $\tau(z_1), \tau(z_2) \in (-1, 1)$.

Proof: Suppose first that (2.6) holds. Then

$$g(z) = \frac{(1 - |z_1|^2) f'(z_1)}{f\left(\frac{z + z_1}{1 + \bar{z}_1 z}\right) - f(z_1)} = \frac{1}{z} + b_0 + b_1 z + \dots \quad (2.8)$$

also satisfies (2.6). It follows from Theorem 2 in [5] that

$$(1 - |z|^2) d_h(0, z)^2 |g'(z)| \leq 1 \quad (2.9)$$

for all $z \in D$. This gives

$$(1 - |z|^2) \frac{(1 - |z_1|^2)^2 |f'(z_1)| \left| f'\left(\frac{z + z_1}{1 + \bar{z}_1 z}\right) \right|}{|1 + \bar{z}_1 z|^2 \left| f\left(\frac{z + z_1}{1 + \bar{z}_1 z}\right) - f(z_1) \right|^2} d_h(0, z)^2 \leq 1.$$

With $z_2 = \frac{z + z_1}{1 + \bar{z}_1 z}$, the above inequality gives (2.7).

The case of equality in (2.7) for $z_1 \neq z_2$ corresponds to the case of equality in (2.9) for $z \neq 0$. As shown in [5] this occurs if and only if g is a rotation of a function of the form $1/L + a$. Hence f is of the form stated.

Let us assume now that (2.7) holds. Then the function g as defined in (2.8) satisfies (2.9). Hence, for $z \in D$ we have

$$(1 - |z|^2) \left(|z| + \frac{1}{3} |z|^3 + \dots \right)^2 \left| -\frac{1}{z^2} + b_1 + \dots \right| \leq 1,$$

which implies that

$$\left(1 - \frac{1}{3} |z|^2 + O(z^3)\right) (1 - \operatorname{Re}\{b_1 z^2\} + O(z^3)) \leq 1$$

as $z \rightarrow 0$. Therefore

$$\operatorname{Re}\left\{b_1 \frac{z^2}{|z|^2}\right\} \leq \frac{1}{3} + O(z)$$

as $z \rightarrow 0$, which in turn gives that $|b_1| \leq 1/3$. Thus

$$(1 - |z_1|^2)^2 |Sf(z_1)| = |Sg(0)| = 6|b_1| \leq 2.$$

Since the point z_1 is arbitrary we conclude that $f \in N$.

Remarks:

1. If $\Omega = f(D)$ is the image domain with Poincaré metric $\lambda(w)|dw|$ and hyperbolic distance δ_h , then (2.7) can be rewritten as

$$\delta_h(w_1, w_2) \leq \sqrt{\lambda(w_1)\lambda(w_2)} |w_1 - w_2|$$

for points $w_1, w_2 \in \Omega$.

2. Theorem 2 resembles a result of Blatter, according to which an analytic function in the unit disc is univalent if and only if

$$|f(z_1) - f(z_2)|^2 \geq \frac{1}{8} \frac{\sinh^2(2d_h(z_1, z_2))}{\cosh(4d_h(z_1, z_2))} \{(1 - |z_1|^2)^2 |f'(z_1)|^2 + (1 - |z_2|^2)^2 |f'(z_2)|^2\}.$$

Here the cases of equality for $z_1 \neq z_2$ only happen when f is of the form $ak \circ \tau + b$ with $\tau(D) = D$, $\tau(z_1), \tau(z_2) \in (-1, 1)$, and k the Koebe function.

3. A DIAMETER BOUND

As was pointed out in the proof of Theorem 2, if $f \in N$ is normalized so that $f(z) = 1/z + b_0 + b_1z + \dots$ then

$$(1 - |z|^2)L^2(|z|)|f'(z)| \leq 1.$$

Lemma 2: Let $f(z) = 1/z + b_0 + b_1z + \dots$ belong to N . Then for $|\zeta| = 1$

$$q(r) = (1 - r^2)L^2(r)|f'(r\zeta)| \tag{3.1}$$

is decreasing for $r \in [0, 1)$.

Proof: Let $|\zeta| = 1$ and for $r \in [0, 1)$ define

$$u(r) = \frac{1}{\sqrt{(1 - r^2)|f'(r\zeta)|}}.$$

It was shown in [5] that

$$\frac{d}{dr}[(1 - r^2)u'(r)] \geq 0. \tag{3.2}$$

Also, $u(0) = 0$ and $u'(0) = 1$. Let

$$v(r) = (1 - r^2)u'(r)L(r) - u(r).$$

A simple calculation shows that

$$v'(r) = L(r) \frac{d}{dr}[(1 - r^2)u'(r)],$$

hence v is increasing. Since $v(0) = 0$ we conclude that $v(r) \geq 0$ for $r \in [0, 1)$, and therefore

$$\frac{d}{dr} \left(\frac{u(r)}{L(r)} \right) = \frac{v(r)}{(1-r^2)L^2(r)} \geq 0.$$

Since $q(r) = (L(r)/u(r))^2$ the lemma follows.

Theorem 3: *Let*

$$f(z) = \frac{1}{z} + b_0 + b_1z + \dots$$

belong to N . Then

$$\text{diam}f(B(z)) \leq K(1 - |z|^2)|f'(z)|L(|z|), \quad (3.3)$$

where K is an absolute constant.

Proof: Let $z = re^{it}$ and $w = \rho e^{i\theta} \in B(z)$. We write $q = (1 - |z|^2)|f'(z)|$. The classical distortion theorems for univalent functions imply that

$$|f(re^{i\theta}) - f(re^{it})| \leq K_1q \quad (3.4)$$

and

$$(1 - r^2)|f'(re^{i\theta})| \leq K_2q, \quad (3.5)$$

where K_1, K_2 are absolute constants. It follows now from Lemma 2 and equation (3.5) that

$$(1 - \rho^2)|f'(\rho e^{i\theta})|L^2(\rho) \leq (1 - r^2)|f'(re^{i\theta})|L^2(r) \leq K_2qL^2(r). \quad (3.6)$$

Hence

$$\begin{aligned} |f(\rho e^{i\theta}) - f(re^{i\theta})| &\leq \int_r^\rho |f'(se^{i\theta})| ds \leq K_2q \int_r^\rho \frac{L^2(r)}{(1-s^2)L^2(s)} ds \\ &\leq K_2qL^2(r) \int_r^1 \frac{L'(s)}{L^2(s)} ds = K_2qL(r). \end{aligned}$$

This inequality together with (3.4) implies (3.3).

In Theorem 3 we have used the stated normalization on f in order to make $\partial f(D)$ bounded. If now $f \in N_0$ then, as pointed out in the introduction, the image Ω will either be a parallel strip or else will be bounded. In the latter, it is clear that (3.3) will still hold with K replaced by some constant M depending on f . From the results in [5], such a domain will be a quasidisc if for some constant M the stronger estimate holds:

$$\text{diam}f(B(z)) \leq M(1 - |z|^2)|f'(z)|,$$

that is, equation (3.3) without the logarithm.

4. BOUNDARY BEHAVIOR AND EXCEPTIONAL POINTS

It was shown in [7] that all Nehari functions admit a (spherically) continuous extension to the closed disc. In this section we shall be interested in studying the behavior of

$$(1 - r^2)\operatorname{Re}\left\{\zeta \frac{f''}{f'}(r\zeta)\right\}$$

as $r \rightarrow 1$. According to Theorem 4 in [5], a Nehari domain is a John domain (hence a quasidisc) if and only if the corresponding function f normalized to be in N_0 satisfies

$$\limsup_{|z| \rightarrow 1} (1 - |z|^2)\operatorname{Re}\left\{z \frac{f''}{f'}(z)\right\} < 2. \quad (4.1)$$

Recall also that if $f \in N_0$ then in any case

$$(1 - |z|^2)\left|\frac{f''}{f'}(z)\right| \leq 2. \quad (4.2)$$

See, e.g., [4].

The following lemma is of a general nature:

Lemma 3: *Let $h(z)$ be analytic in D and suppose that for some $0 < \alpha < \infty$, $M < \infty$*

$$(1 - |z|)^\alpha |g(z)| \leq M. \quad (4.3)$$

Then there exist at most countably many points ζ , $|\zeta| = 1$, such that

$$\lim_{r \rightarrow 1} (1 - r)^\alpha g(r\zeta) =: b(\zeta) \neq 0 \text{ exists.} \quad (4.4)$$

Proof: Let $|\zeta| = 1$ be such that (4.4) holds. For $r \in (0, 1)$ and $z \in D$ let

$$f(z, r) = \left(1 - \frac{z+r}{1+rz}\right)^\alpha g\left(\frac{z+r}{1+rz}\zeta\right). \quad (4.5)$$

Hence by (4.3)

$$\begin{aligned} |f(z, r)| &\leq 2^\alpha M \left|1 - \frac{z+r}{1+rz}\right|^\alpha \left(1 - \left|\frac{z+r}{1+rz}\right|^2\right)^{-\alpha} \\ &= \frac{2^\alpha M |1+rz|^\alpha |1-z|^\alpha}{(1+r)^\alpha (1-|z|^2)^\alpha} \leq 4^\alpha M \left(\frac{|1-z|}{1-|z|}\right)^\alpha. \end{aligned} \quad (4.6)$$

Therefore as $r \rightarrow 1$, $f(z, r)$ is locally uniformly bounded in z . Also, by (4.4) and (4.5) we have

$$f(z, r) = (1 - \bar{\zeta}w)^\alpha g(w) \rightarrow b(\zeta) \quad (4.7)$$

as $r \rightarrow 1$, where $w = \zeta \frac{z+r}{1+rz}$. From the theorem of Montel we conclude that (4.7) holds locally uniformly in z .

Let

$$\varphi(z) = (1 - |z|^2)^\alpha |g(z)|.$$

If ζ is such that (4.4) holds, then according to (4.5)

$$\varphi\left(\frac{z+r}{1+rz}\zeta\right) = \frac{(1-r^2)^\alpha(1-|z|^2)^\alpha}{|1+rz|^{2\alpha}} \left|g\left(\frac{z+r}{1+rz}\zeta\right)\right| \rightarrow \left(\frac{1-|z|^2}{|1-z|}\right)^\alpha |b(\zeta)|$$

as $r \rightarrow 1$. The convergence is locally uniform in z . Hence as $z \rightarrow \zeta$ radially

$$\varphi(z) \rightarrow 2^\alpha |b(\zeta)| \neq 0,$$

but we can also find a curve γ ending at ζ along which the function φ tends to 0. The Ambiguous Point Theorem of Bagemihl, [1], implies that the number of points ζ for which this can happen is at most countable. This finishes the proof.

Theorem 4: *Let $f \in N_0$. Then*

$$\liminf_{r \rightarrow 1} (1-r^2) \operatorname{Re}\left\{\zeta \frac{f''}{f'}(r\zeta)\right\} < 2, \quad (4.8)$$

except possibly for countably many points ζ .

Proof: Observe first that in light of (4.2), the quantity on the left hand side of (4.8) is bounded by 2. Suppose ζ is such that we have equality in (4.8). Then

$$\lim_{r \rightarrow 1} (1-r^2) \operatorname{Re}\left\{\zeta \frac{f''}{f'}(r\zeta)\right\} = 2.$$

Using (4.2) again we conclude that

$$\lim_{r \rightarrow 1} (1-r^2) \operatorname{Im}\left\{\zeta \frac{f''}{f'}(r\zeta)\right\} = 0,$$

hence

$$\lim_{r \rightarrow 1} (1-r^2) \frac{f''}{f'}(r\zeta) = 2.$$

But according to Lemma 3 applied to $g = f''/f'$ and $\alpha = 1$, the last equation can only happen for countably many points ζ .

Lemma 4: *Let*

$$f(z) = \frac{1}{z} + b_0 + b_1z + \dots$$

be in N . Then for $|\zeta| = 1$ and $r \in (0, 1)$

$$|f(\zeta) - f(r\zeta)| \leq \int_r^1 |f'(s\zeta)| ds \leq \frac{(1-r^2)|f'(r\zeta)|}{r - \frac{1-r^2}{2} \operatorname{Re}\{\zeta \frac{f''}{f'}(r\zeta)\}}. \quad (4.9)$$

Remark:

The right-hand side of (4.9) is rather similar to the extension operator considered in [4].

Proof: Let $|\zeta| = 1$ and let again

$$u(r) = \frac{1}{\sqrt{(1-r^2)|f'(r\zeta)|}}.$$

Then $((1-r^2)u'(r))' \geq 0$, hence for $0 < r < s < 1$

$$(1-s^2)u'(s) \geq (1-r^2)u'(r),$$

and therefore

$$u(s) - u(r) \geq (1-r^2)u'(r)(L(s) - L(r)).$$

Thus

$$(1-s^2)|f'(s\zeta)| = \frac{1}{u^2(s)} \leq \frac{1}{u^2(r)[1 + (1-r^2)\frac{u'}{u}(r)(L(s) - L(r))]^2},$$

or

$$\begin{aligned} |f'(s\zeta)| &\leq \frac{(1-r^2)|f'(r\zeta)|L'(s)}{[1 + (1-r^2)\frac{u'}{u}(r)(L(s) - L(r))]^2} \\ &= \frac{|f'(r\zeta)|}{(u'/u)(r)} \frac{d}{ds} \left[\frac{-1}{1 + (1-r^2)\frac{u'}{u}(r)(L(s) - L(r))} \right]. \end{aligned}$$

This implies

$$\int_r^1 |f'(s\zeta)| ds \leq \frac{|f'(r\zeta)|}{(u'/u)(r)},$$

which is equivalent to (4.9) since

$$(1-r^2)\frac{u'}{u}(r) = r - \frac{1-r^2}{2} \operatorname{Re}\{\zeta \frac{f''}{f'}(r\zeta)\}.$$

Theorem 5: *Let $f \in N$. Then*

$$\liminf_{r \rightarrow 1} \frac{|f(\zeta) - f(r\zeta)|}{(1 - r^2)|f'(r\zeta)|} < \infty \quad (4.10)$$

with the exception of at most countably many points $\zeta \in \partial D$.

Proof: Without loss of generality we may assume that

$$f(z) = \frac{1}{z} + b_0 + b_1z + \cdots,$$

since such a normalization can affect condition (4.10) at most at one boundary point. Also, we may take $b_0 = 0$. Then (4.10) follows directly from (4.8) and (4.9) provided we can show that (4.8) still holds when $f(z) = 1/z + \cdots$. Let $g = 1/f$. Then $g \in N_0$, hence it is either a rotation of L or else it is bounded. In the latter, it is easy to see that (4.8) for g implies (4.8) for f , while if $g = 1/L$ then (4.8) can be verified directly.

To conclude, we remark that if $f(D)$ is a bounded quasidisc then

$$\limsup_{r \rightarrow 1} \frac{|f(\zeta) - f(r\zeta)|}{(1 - r^2)|f'(r\zeta)|} < \infty \quad (4.11)$$

for all ζ . It is natural to ask whether a stronger form of Theorem 5 is true, where (4.11) holds with at most countably many exceptions.

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